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# Euler's factorial series and global relations

Tapani Matala-aho<sup>1</sup> and Wadim Zudilin<sup>2,3</sup>

<sup>1</sup>MATEMATIIKKA, PL 3000, 90014 OULUN YLIOPISTO, FINLAND

<sup>2</sup>INSTITUTE FOR MATHEMATICS, ASTROPHYSICS AND PARTICLE PHYSICS,  
Radboud Universiteit, PO Box 9010, 6500 GL Nijmegen, The Netherlands

<sup>3</sup>SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES,  
THE UNIVERSITY OF NEWCASTLE, CALLAGHAN, NSW 2308, AUSTRALIA

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*To the memory of Marc Huttner (1947–2015)*

## Abstract

Using Padé approximations to the series  $E(z) = \sum_{k=0}^{\infty} k!(-z)^k$ , we address arithmetic and analytical questions related to its values in both  $p$ -adic and Archimedean valuations.

## 1 Introduction

In his 1760 paper on divergent series [5], L. Euler introduced and studied the formal (hypergeometric) series

$$E(z) := \sum_{k=0}^{\infty} k!(-z)^k \quad (1)$$

(see also [1, 2] and, in particular, [8, Section 2.5]), which specializes at  $z = 1$  to Wallis' series

$$W := \sum_{k=0}^{\infty} (-1)^k k!, \quad (2)$$

which has teased people's imagination since the times of Euler. Notice that the series (1) is a perfectly convergent  $p$ -adic series in the disc  $|z|_p \leq 1$  for all primes  $p$ , so that one can discuss the arithmetic properties of its values, for example, at  $z = 1$  (the Wallis case) and at  $z = -1$ . The irrationality of the latter specialization,

$$K = K_p := \sum_{k=0}^{\infty} k!, \quad (3)$$

for any prime  $p$  is a folklore conjecture [12, p. 17] (see [11], also for a link of the problem to a combinatorial conjecture of H. Wilf). Already the expectation  $|K_p|_p = 1$  (so that  $K_p$  is a  $p$ -adic unit) for all primes  $p$ , which is an equivalent form of Kurepa's conjecture [7] from 1971, remains open; see [4] for the latest achievements in this direction.

In what follows, we refer to the series in (1) as Euler's factorial series and label it  $E_p(z)$  when treat it as the function in the  $p$ -adic domain for a given prime  $p$ .

**Theorem 1.** *Given  $\xi \in \mathbb{Z} \setminus \{0\}$ , let  $\mathcal{P}$  be a subset of prime numbers such that*

$$\limsup_{n \rightarrow \infty} c^n n! \prod_{p \in \mathcal{P}} |n!|_p^2 = 0, \quad \text{where } c = c(\xi; \mathcal{P}) := 4|\xi| \prod_{p \in \mathcal{P}} |\xi|_p^2. \quad (4)$$

*Then either there exists a prime  $p \in \mathcal{P}$  for which  $E_p(\xi)$  is irrational, or there are two distinct primes  $p, q \in \mathcal{P}$  such that  $E_p(\xi) \neq E_q(\xi)$  (while  $E_p(\xi), E_q(\xi) \in \mathbb{Q}$ ).*

Because  $\prod_p |n!|_p = 1/n!$  when the product is taken over all primes, condition (4) is clearly satisfied for any subset  $\mathcal{P}$  whose complement in the set of all primes is finite. This also suggests more exotic choices of  $\mathcal{P}$ . Furthermore, the conclusion of the theorem is contrasted with Euler's sum

$$\sum_{k=0}^{\infty} k \cdot k! = -1,$$

which is valid in any  $p$ -adic valuation (and follows from  $\sum_{k=0}^{n-1} k \cdot k! = n! - 1$ ) — an example of what is called a *global* relation.

The idea behind the proof of Theorem 1 is to construct approximations  $p_n/q_n$  to the number  $\omega_p = E_p(\xi)$  in question, which do not depend on  $p$  and approximate the number considerably well for each  $p$ -adic valuation:  $q_n \omega_p - p_n = r_{n,p}$  for  $n = 0, 1, 2, \dots$ ;  $|r_{n,p}|_p \rightarrow 0$  as  $n \rightarrow \infty$  and there are infinitely many indices  $n$  for which  $r_{n,p} \neq 0$  for at least one prime  $p \in \mathcal{P}$ . Assume that  $\omega_p = a/b$ , the same rational number, for all  $p \in \mathcal{P}$ . Then  $q_n a - p_n b \in \mathbb{Z} \setminus \{0\}$ , so that  $0 < |q_n a - p_n b|_p \leq 1$ , for infinitely many indices  $n$  and *all* primes  $p$ , hence

$$\begin{aligned} 1 &= |q_n a - p_n b| \prod_p |q_n a - p_n b|_p \leq |q_n a - p_n b| \prod_{p \in \mathcal{P}} |q_n a - p_n b|_p \\ &\leq (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |b r_{n,p}|_p \\ &\leq (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |r_{n,p}|_p \end{aligned}$$

for those  $n$ . This means that the condition

$$\limsup_{n \rightarrow \infty} \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |r_{n,p}|_p = 0 \quad (5)$$

contradicts the latter estimate, thus making the  $\mathcal{P}$ -global linear relation  $\omega_p = a/b$  impossible.

The result in Theorem 1 can be put in a general context of global relations for Euler-type series; the corresponding settings can be found in the paper [3]. We do not pursue this route here as our principal motivation is a sufficiently elementary arithmetic treatment of an analytical function that have some historical value. The rational approximations to  $E(z)$  we construct in Section 2 are Padé approximations; in spite of being known for centuries, implicitly from the continued fraction for  $E(z)$  given by Euler himself [5] and explicitly from the work of T. J. Stieltjes [13], these Padé approximations remain a useful source for arithmetic and analytical investigations. In Section 3 we revisit Euler's summation of Wallis' series (2) using the approximations; we also complement the derivation by providing 'Archimedean analogue(s)' for the divergent series  $K = E(-1)$  from (3) — the case when the classical strategy does not work.

Our way of constructing the Padé approximations is inspired by a related Padé construction of M. Hata and M. Huttner in [6]. The construction was a particular favourite of Marc Huttner who, for the span of his mathematical life, remained a passionate advocate of interplay between Picard–Fuchs linear differential equations and Padé approximations. We dedicate this work to his memory.

## 2 Hypergeometric series and Padé approximations

Euler's factorial series (1) is the particular  $a = 1$  instance of the hypergeometric series

$${}_2F_0(a, 1 \mid z) = \sum_{k=0}^{\infty} (a)_k z^k. \quad (6)$$

Here and in what follows, we use the Pochhammer notation  $(a)_k$  which is defined inductively by  $(a)_0 = 1$  and  $(a)_{k+1} = (a+k)(a)_k$  for  $k \in \mathbb{Z}_{\geq 0}$ . Our Padé approximations below are given more generally for the function (6).

**Theorem 2.** *For  $n, \lambda \in \mathbb{Z}_{\geq 0}$ , take*

$$B_{n,\lambda}(z) = \sum_{i=0}^n \binom{n}{i} \frac{(-1)^i z^{n-i}}{(a)_{i+\lambda}}.$$

*Then  $\deg_z B_{n,\lambda} = n$  and for a polynomial  $A_{n,\lambda}(z)$  of degree  $\deg_z A_{n,\lambda} \leq n + \lambda - 1$  we have*

$$B_{n,\lambda}(z) {}_2F_0(a, 1 \mid z) - A_{n,\lambda}(z) = L_{n,\lambda}(z), \quad (7)$$

*where  $\text{ord}_{z=0} L_{n,\lambda}(z) = 2n + \lambda$ . Explicitly,*

$$L_{n,\lambda}(z) = (-1)^n n! z^{2n+\lambda} \sum_{k=0}^{\infty} k! \binom{n+k}{k} \binom{n+k+a+\lambda-1}{k} z^k. \quad (8)$$

*Proof.* Relation (7) means that there is a 'gap' of length  $n$  in the power series expansion

$$B_{n,\lambda}(z) {}_2F_0(a, 1 \mid z) = A_{n,\lambda}(z) + L_{n,\lambda}(z).$$

Write  $B_{n,\lambda}(z) = \sum_{h=0}^n b_h t^h$  and consider the series expansion of the product

$$B_{n,\lambda}(z) {}_2F_0(a, 1 \mid z) = \sum_{l=0}^{\infty} r_l z^l,$$

where  $r_l = \sum_{h+k=l} b_h(a)_k$ ; in particular,

$$r_l = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(a)_{i+\lambda+m}}{(a)_{i+\lambda}} = \sum_{i=0}^n (-1)^i \binom{n}{i} (a+i+\lambda)_m$$

with  $m = l - n - \lambda$  for  $l > n + \lambda - 1$ . To verify the desired ‘gap’ condition,

$$r_{n+\lambda} = r_{n+\lambda+1} = \cdots = r_{n+\lambda+n-1} = 0, \quad (9)$$

we introduce the shift operators  $N = N_a$  and  $\Delta = \Delta_a = N - \text{id}$  defined on functions  $f(a)$  by

$$Nf(a) = f(a+1) \quad \text{and} \quad \Delta f(a) = f(a+1) - f(a).$$

It follows from

$$\Delta^n(a+\lambda)_m = (-1)^n (-m)_n (a+\lambda+n)_{m-n} \quad \text{for } n \in \mathbb{Z}_{\geq 0}$$

that

$$\begin{aligned} r_l &= \sum_{i=0}^n (-1)^i \binom{n}{i} N^i(a+\lambda)_m = (\text{id} - N)^n(a+\lambda)_m \\ &= (-\Delta)^n(a+\lambda)_m = (-m)_n (a+\lambda+n)_{m-n}, \end{aligned}$$

which, in turn, implies (9) because of the vanishing of  $(-m)_n$  for  $m = 0, 1, \dots, n-1$ . The explicit expression for  $r_l$  just found also gives

$$r_{2n+\lambda+k} = (-n-k)_n (a+\lambda+n)_k = (-1)^n n! \binom{n+k}{k} k! \binom{n+k+a+\lambda-1}{k},$$

hence the closed form (8). We also have

$$A_{n,\lambda}(z) = \sum_{l=0}^{n+\lambda-1} r_l z^l = \sum_{l=0}^{n+\lambda-1} z^l \sum_{\substack{i=0 \\ i \geq n-l}}^n (-1)^i \binom{n}{i} \frac{(a)_{i+l-n}}{(a)_{i+\lambda}}.$$

This concludes our proof of the theorem.  $\square$

From now on we choose  $a = 1$ ,  $\lambda = 0$ , change  $z$  to  $-z$  and renormalize the corresponding Padé approximations produced by Theorem 2 by multiplying them by

$(-1)^n n! :$

$$Q_n(z) := (-1)^n n! B_{n,0}(-z) = \sum_{i=0}^n \binom{n}{i} \frac{n!}{i!} z^{n-i} = \sum_{i=0}^n i! \binom{n}{i}^2 z^i,$$

$$P_n(z) := (-1)^n n! A_{n,0}(-z) = (-1)^n \sum_{l=0}^{n-1} (-z)^l \sum_{i=n-l}^n (-1)^{n-i} \binom{n}{i} \frac{n! (i+l-n)!}{i!}$$

$$= (-1)^n \sum_{l=0}^{n-1} (-z)^l \sum_{i=0}^l (-1)^i i! (l-i)! \binom{n}{i}^2$$

and

$$R_n(z) := (-1)^n n! L_{n,0}(-z) = n!^2 z^{2n} \sum_{k=0}^{\infty} (-1)^k k! \binom{n+k}{k}^2 z^k.$$

In this notation, the Padé approximation formula (7) may be rewritten as

$$Q_n(z)E(z) - P_n(z) = R_n(z) \quad (10)$$

for  $n \in \mathbb{Z}_{>0}$ . Observe that

$$Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = n!^2 z^{2n}. \quad (11)$$

Indeed, the standard Padé walkabout proves the identity

$$Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = Q_{n+1}(z)R_n(z) - Q_n(z)R_{n+1}(z),$$

in which the degree of the left-hand side is at most  $2n$  while the order of the right-hand side is at least  $2n$ .

*Proof of Theorem 1.* We may use the formal series identity (10) to get appropriate numerical approximations for any at  $z = \xi \in \mathbb{Z} \setminus \{0\}$ . For  $n = 1, 2, \dots$ , take  $p_n = P_n(\xi) \in \mathbb{Z}$ ,  $q_n = Q_n(\xi) \in \mathbb{Z}$  and define

$$r_{n,p} = R_n(\xi) = q_n E_p(\xi) - p_n$$

for each prime  $p \in \mathcal{P}$ . Using elementary summation formulas and trivial estimates for binomials we have

$$|q_n| \leq |\xi|^n n! \sum_{i=0}^n \binom{n}{i}^2 = |\xi|^n n! \binom{2n}{n} < 4^n |\xi|^n n!,$$

$$|p_n| \leq |\xi|^{n-1} n \sum_{i=0}^{n-1} i! (n-1-i)! \binom{n}{i}^2 \leq |\xi|^n n! \sum_{i=0}^n \binom{n}{i}^2 < 4^n |\xi|^n n!$$

and

$$|r_{n,p}|_p = |\xi|_p^{2n} |n!|_p^2 \left| \sum_{k=0}^{\infty} (-1)^k k! \binom{n+k}{k}^2 \xi^k \right|_p \leq |\xi|_p^{2n} |n!|_p^2.$$

Therefore, condition (5) reads

$$\limsup_{n \rightarrow \infty} 4^n |\xi|^n n! \prod_{p \in \mathcal{P}} |\xi|_p^{2n} |n!|_p^2 = 0$$

and because for at least one  $p \in \mathcal{P}$  we have either  $r_{n,p} \neq 0$  or  $r_{n+1,p} \neq 0$  from (11), the theorem follows.  $\square$

### 3 Summation of divergent series

Fix a prime  $p$ . If we restrict, for simplicity, the sequence of indices  $n$  to the arithmetic progression  $n \equiv 0 \pmod{p}$ , then it is not hard to see from the calculation in Section 2 that the sequence of rational approximations  $p_n/q_n = P_n(\xi)/Q_n(\xi)$  converges  $p$ -adically to  $E(\xi)$ . Indeed,

$$q_n = 1 + \sum_{i=1}^{p-1} i! \binom{n}{i}^2 \xi^i + \sum_{i=p}^n i! \binom{n}{i}^2 \xi^i \equiv 1 \pmod{p}$$

is a  $p$ -adic unit (all the binomials are divisible by  $p$  in the first sum, while  $i!$  is divisible by  $p$  in the second one), and

$$\left| E(\xi) - \frac{p_n}{q_n} \right|_p = |q_n|_p^{-1} |q_n E(\xi) - p_n|_p = |r_{n,p}|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

When  $\xi > 0$ , the same sequence of rational numbers  $p_n/q_n = P_n(\xi)/Q_n(\xi)$  converges to the value at  $z = \xi$  of the integral

$$\tilde{E}(z) = \int_0^\infty \frac{e^{-s}}{1+zs} ds \quad (12)$$

with respect to the Archimedean norm. The integral itself converges for any  $z \in \mathbb{C} \setminus (-\infty, 0]$ , though its formal Taylor expansion at  $z = 0$  (obtained by expanding  $1/(1+zs)$  into the power series under the integral sign) is precisely Euler's factorial series  $E(z)$  as in (1). In particular, relation (10) remains valid with  $E(z)$  replaced by  $\tilde{E}(z)$  and  $R_n(z)$  by

$$\tilde{R}_n(z) = n! z^{2n} \int_0^\infty \frac{s^n e^{-s}}{(1+zs)^{n+1}} ds \quad \text{for } n = 0, 1, 2, \dots,$$

so that

$$\left| \tilde{E}(\xi) - \frac{p_n}{q_n} \right| = \frac{\tilde{R}_n(\xi)}{Q_n(\xi)} \leq \frac{n^n}{(n+1)^{n+1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where the estimates  $Q_n(\xi) \geq n! \xi^n$  and

$$\tilde{R}_n(\xi) \leq n! \xi^{2n} \left( \max_{s>0} \frac{s^n}{(1+\xi s)^{n+1}} \right) \int_0^\infty e^{-s} ds = n! \xi^n \frac{n^n}{(n+1)^{n+1}}$$

were used. This computation reveals us that

$$\lim_{n \rightarrow \infty} \frac{p_n}{q_n} = \tilde{E}(\xi) = -xe^x \left( \gamma + \log x + \sum_{k=1}^{\infty} \frac{(-x)^k}{k \cdot k!} \right) \Big|_{x=1/\xi},$$

where  $\gamma = 0.5772156649 \dots$  is Euler's constant. In particular,

$$W = \tilde{E}(1) = e \left( -\gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \cdot k!} \right) = 0.5963473623 \dots$$

for Wallis' series (2). The resulted quantity is known as the Euler–Gompertz constant [8].

The strategy in the previous paragraph does not apply to  $z = \xi < 0$ , somewhat already observed by Stieltjes in [13]. The analytical continuation of the the function  $\tilde{E}(z)$  depends on whether we perform the integration along the upper or lower banks of the ray  $[0, \infty)$  in (12); denote the corresponding values by  $\tilde{E}_+(z)$  and  $\tilde{E}_-(z)$ , respectively. By considering the integration of  $e^{-s}/(1+zs)$  along the curvilinear triangle that consists of the segment  $[0, R]$  (along a particular bank), the arc  $[R, R e^{\sqrt{-1}\theta}]$  followed by the segment  $[R e^{\sqrt{-1}\theta}, 0]$ , where  $0 < \theta < \pi/2$  for the upper bank and  $-\pi/2 < \theta < 0$  for the lower one, and then taking the limit as  $R \rightarrow \infty$  (so that the integral along the arc tends to 0) we conclude that

$$\tilde{E}_{\pm}(z) = \int_0^{e^{\sqrt{-1}\theta}\infty} \frac{e^{-s}}{1+zs} ds = -xe^x \left( \gamma + \log |x| \pm \sqrt{-1}\pi + \sum_{k=1}^{\infty} \frac{(-x)^k}{k \cdot k!} \right) \Big|_{x=1/z}, \quad (13)$$

with the choice of  $\theta$  arbitrary in the interval  $0 < \theta < \pi/2$  for  $\tilde{E}_+(z)$  and in the interval  $-\pi/2 < \theta < 0$  for  $\tilde{E}_-(z)$ . In particular,

$$\begin{aligned} K = \tilde{E}_{\pm}(-1) &= \frac{1}{e} \left( \gamma + \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} \mp \sqrt{-1}\pi \right) \\ &= 0.6971748832 \dots \mp \sqrt{-1} \cdot 1.1557273497 \dots \end{aligned}$$

for the series in (3).

## 4 Final remarks

In [14] we outline a different strategy of proving a result analogous to Theorem 1 on using the Hankel determinants generated by the tails of Euler's factorial series (1). As the condition on a subset of primes  $\mathcal{P}$  in that result is spiritually similar to (4),



we do not detail the derivation here. However we stress that a potential combination of the two methods, namely, using the Hankel determinants generated by the Padé approximations of Euler’s factorial series, may be a source of further novelties on the topic. A discussion on this type of construction in the Archimedean setting can be found in [15].

One consequence of the formula in (13), which uncovers a pair of *complex* conjugate values for  $E(\xi)$  when  $\xi < 0$ , is that the *rational* approximations  $p_n/q_n = P_n(\xi)/Q_n(\xi)$  do not converge at all in such cases. Interestingly enough, the Hankel determinants ‘see’ those complex values (13) as experimentally observed in [14].

Finally, we would like to note that nothing is known about the irrationality and transcendence of the Archimedean valuations of Euler’s factorial series (1) at rational  $z = \xi$  (see the discussion in [8, Sections 3.15, 3.16]). This is in contrast with its  $q$ -analogue

$$\sum_{k=0}^{\infty} z^k \prod_{i=1}^k (1 - q^i),$$

for which irrationality and linear independence results are known in Archimedean and non-Archimedean places alike — see [9]. Further details on a nice  $q$ -counterpart of the Padé approximation analysis can be found in [10].

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